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ASYMPTOTIC MODELING OF NONLINEAR WAVE PROCESSES IN SHOCK-LOADED ELASTOPLASTIC MATERIALS

N. N. Myagkov

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Nonlinear wave processes in shock-loaded elastoplastic materials are modeled. A comparison of the results obtained with experimental data and numerical solutions of exact systems of dynamic equations shows that the model equations proposed qualitatively describe the stress-distribution evolution in both the elastic-flow and plastic-flow regions and can be used to solve one- and two-dimensional problems of pulsed deformation and fracture of elastoplastic media.

Key words: *nonlinear waves, wave interaction, shock-loaded materials, elastoplastic materials.*

Introduction. Studying the evolution of nonlinear waves generated by an explosion or shock loading of materials and structures is of scientific and practical interest. Experiments of this kind are usually performed under conditions of pulsed loading produced by a contact explosion of explosives or by an impact with velocities up to 2 km/sec, which corresponds to the stress range from several pascals to tens of gigapascals. In these cases, the stress amplitude is usually smaller than the bulk modulus (but much higher than the elastic limit); therefore, asymptotic methods can be used to model shock-wave processes. In most cases, apparently, asymptotic models are the only tool to perform analytical studies and construct simplified numerical algorithms.

1. Asymptotic Models of Nonlinear Longitudinal Waves Propagating in Elastoplastic Media. Myagkov [1, 2] proposed an asymptotic model of propagation of nonlinear longitudinal waves in elastoplastic media. The equations of the model have the form [1, 2]

$$\lambda_i \frac{\partial V_i}{\partial z} - \frac{1}{4}(\alpha+2)V_i \frac{\partial V_i}{\partial \xi_i} - 3\nu \frac{\partial \psi_i}{\partial \xi_i} - \frac{1}{2}\mu \frac{\partial^2 V_i}{\partial \xi_i^2} - \frac{1}{2}\varepsilon_\Delta \int\limits_{-\infty}^{\xi_i} \Delta_\perp V_i \, d\xi_i' = 0. \qquad (1)$$

Here $V_i = -\sigma_{11}' + \lambda_i u_1'$, $\sigma_{11}' = \sigma_{11}/(\rho_0 C_0^2)$, $u_1' = u_1/C_0$, $\lambda_{1,2} = \pm 1$, $z = x_1'(1 + O(\varepsilon))$ is the dimensionless Lagrange coordinate in the x_1 direction, $\xi_i = t' - \lambda_i^{-1} x_1'$ is the phase variable, $t' = t/t_0$, $x_1' = x_1/(C_0 t_0)$, t_0 is the characteristic time, C_0 is the volume speed of sound, $\Delta_\perp = \partial^2/\partial r'^2 + (1/r')\partial/\partial r'$ ($r' = \sqrt{x_2^2 + x_3^2}/r_0$), σ_{11} and u_1 are the components of the stress tensor and velocity vector of the medium, ρ_0 is the unperturbed density, ε, ε_Δ, ν, and μ are small parameters: ε is determined as the ratio of the stress amplitude to the bulk modulus and ε_Δ characterizes the transverse divergence (wave diffraction) and is determined as the squared ratio of the wavelength to the linear dimension of the loading region, $\nu = (C_{\rm long}^2 - C_0^2)/(2C_0^2)$ ($C_{\rm long}$ is the phase velocity of longitudinal elastic waves), μ characterizes the internal-friction viscosity and thermal conductivity, and the parameter α is determined from the equation of state.

Equations (1) are closed by the constitutive equation of the medium, which gives a functional relation between $\psi_i(z, \xi_i)$ and $V_i(z, \xi_i)$. This relation can be written as $\psi_i = \hat{R}[V_i]$, where $\hat{R}[V_i]$ is the result of the action of the

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nonlinear operator specified implicitly by the solution of the constitutive equation. To ensure uniform applicability of Eqs. (1) for $z \leqslant O[\min\{\varepsilon^{-1}, \mu^{-1}, \nu^{-1}, \varepsilon_\Delta^{-1}\}]$, it is sufficient to satisfy the conditions

$$|J_i^{(\varepsilon)}| = O(\varepsilon), \qquad |J_i^{(\nu)}| = O(\varepsilon), \tag{2}$$

where

$$J_i^{(\varepsilon)} = \int\limits_{-\infty}^{\xi_j} V_j\, d\xi_j; \quad J_i^{(\nu)} = \int\limits_{-\infty}^{\xi_j} \Big(\frac{\partial \psi}{\partial \xi_i} - \frac{\partial \psi_i}{\partial \xi_i}\Big) d\xi_j; \quad \psi = \hat{R}[V_1 + V_2]; \quad \xi_j \in (-\infty, +\infty);$$

$i, j = 1, 2$ $(i \neq j)$. The first condition in (2) is satisfied by specifying appropriate boundary conditions. The form of the integrals $J_i^{(\nu)}$ implies that the second condition in (2) should hold, at least, for rapidly decaying solutions V_j as $|\xi_j| \to \infty$. Generally, whether this condition is satisfied can be verified only for the known functions V_j, i.e., when the problem is solved.

Deriving Eqs. (1), one can obtain "self-consistent" expressions that give the velocity-stress relations with accuracy to terms of the second order of smallness inclusively:

$$\begin{aligned} \sigma'_{11} &= -\lambda_i u'_1 - \frac{1}{4}\,(\alpha+2)u_1'^2 + \frac{1}{2}\,\mu\lambda_i\,\frac{\partial u'_1}{\partial \xi_i} - \frac{3}{2}\,\nu\psi, \\ \lambda_i u'_1 &= -\sigma'_{11} - \frac{1}{4}\,(\alpha+2)\sigma_{11}'^2 - \frac{1}{2}\,\mu\,\frac{\partial \sigma'_{11}}{\partial \xi_i} - \frac{3}{2}\,\nu\psi. \end{aligned} \tag{3}$$

Expressions (3) generalize the relations known in nonlinear acoustics [3] to the case of an inelastic medium.

To take into account the interaction between nonlinear waves that refer to different characteristic directions, in contrast to (1), the phase variables should be written in the form [1, 2]

$$\xi_i = t' - \lambda_i^{-1}(x'_1 + \varepsilon\Phi_i(x'_1, t') + \nu\theta_i(x'_1, t')), \qquad i = 1, 2, \tag{4}$$

i.e., corrections of order ε and ν are introduced into the phase variables. As a result, the starting system of equations is again reduced to Eqs. (1), constitutive equation, and equations for the phase functions:

$$\begin{aligned} \varepsilon\,\frac{\partial \Phi_i}{\partial z} &= -\frac{1}{4}\,(\alpha+2)V_j, \\ \frac{\partial \theta_i}{\partial z} &= -3\Big(\frac{\partial\psi/\partial\xi_i}{\partial V/\partial\xi_i} - \frac{\partial\psi_i/\partial\xi_i}{\partial V_i/\partial\xi_i}\Big)_z, \qquad i \neq j, \quad i, j = 1, 2. \end{aligned} \tag{5}$$

Here $V = V_1 + V_2$. We note that the wave interaction is taken into account implicitly by nonuniform deformation of the phase variables in the solution that ignores the interaction.

Introduction of corrections of order ε and ν into the phase variables extends the range of applicability of the solutions of Eqs. (1) to the case of extended waves. In this case, the solutions of Eqs. (1) yield the first approximation, which is uniformly applicable for $z \leqslant O[\min\{\varepsilon^{-1}, \mu^{-1}, \nu^{-1}, \varepsilon_\Delta^{-1}\}]$.

It is worth noting that the constitutive equation enters into system (1), which describes quasiplane waves, in the same manner as into the system of equations for plane waves. In solving particular problems, it is necessary to specify the form of the dependence $\psi_i = \hat{R}[V_i]$. For example, if dislocation relations are used for this purpose, it is convenient to write the constitutive equation in the form proposed by Malvern and Duval [4], which can be written in our notation as

$$\frac{\partial \psi_i}{\partial \xi_i} = \frac{1}{3}\,\frac{\partial V_i}{\partial \xi_i} + \frac{4}{3}\,\frac{\partial \varepsilon^{\rm pl}[V_i, \psi_i]}{\partial \xi_i} \qquad (i = 1, 2), \tag{6}$$

where $\varepsilon^{\rm pl}$ is the plastic shear strain. Based on the dislocation concepts, the plastic-strain rate of polycrystalline solids is usually determined by the Orowan relation [5].

The dependences $\psi = \psi(V)$ closing Eqs. (1) can be specified using the constitutive relations of the theory of small elastoplastic strains [6]. If an elastoplastic-flow model of the Prandtl–Reuss type [6, 7] is used to describe the plastic behavior of a medium, the constitutive equation can be written as

$$\frac{\partial \psi_i}{\partial \xi_i} = \frac{1}{3}\,\frac{\partial V_i}{\partial \xi_i} - \bar{\lambda}\psi_i, \qquad i = 1, 2.$$

The parameter $\bar{\lambda}$ is determined from the Mises yield criterion, which has the form $|\psi_i| \leqslant Y/(3G)$ for uniaxial strain (Y is the tensile yield point and G is the shear modulus). In the general case, the yield point is a function of pressure, temperature, plastic strains, strain rates, and other parameters of state.

Using Eqs. (3), one can easily estimate the residual mass velocity in the central cross sections of the specimen after passage of a localized load pulse for which the normal stress tends to zero as $\xi_1 \to \pm\infty$. The residual velocity occurs due to the hysteresis of the elastoplastic-deformation cycle and is given by $u_1' = -(3/2)\nu\psi_1(+\infty)$, where $\psi_1(+\infty) = -Y/(3G)$ [the velocity in (3) was estimated for $\mu' = 0$]. The residual mass velocity was observed experimentally, for example, in [8].

Using the variables $(t', \tilde{\xi}_i)$, we write the approximate system of equations similar to (1):

$$\frac{\partial V_i}{\partial t'} + \frac{1}{4}\lambda_i(\alpha+2)V_i\frac{\partial V_i}{\partial \tilde{\xi}_i} + 3\lambda_i\nu\frac{\partial \psi_i}{\partial \tilde{\xi}_i} - \frac{1}{2}\mu\frac{\partial^2 V_i}{\partial \tilde{\xi}_i^2} + \frac{1}{2}\varepsilon_\Delta\int\limits_{-\infty}^{\tilde{\xi}_i}\Delta_\perp V_i\,d\xi_i' = 0. \tag{1'}$$

Here $\tilde{\xi}_i = x_1' - \lambda_i t'$ ($i = 1, 2$). In this case, the interaction of nonlinear waves corresponding to different characteristic directions is also taken into account by expressing the phase variables in the form (4) and subsequent writing equations for the phase functions.

2. Plane Problems of Propagation and Interaction of Shock Waves. 2.1. *Propagation of a Shock Pulse Induced by Plate Collision.* We consider the normal impact of a plate of thickness l with a velocity $u_0 > 0$ ($u_0/C_0 \sim \varepsilon \ll 1$) upon the surface of a quiescent target plate of thickness L at the moment $t = 0$. Since the in-plane dimensions of the plates are much greater than their thickness, the problem can be treated in one-dimensional formulation. We assume that the impactor and target are made of the same material (aluminum).

We consider the wave evolution V_1 in the target under the assumption that l is small compared to L, which is typical of experiments. In this case, the boundary condition at the free surface is written as

$$V_1 = \begin{cases} u_0/C_0, & 0 < t \leqslant t_1,\ x_1 = -l + u_0 t, \\ 0, & t > t_1,\ x_1 = -l + u_0 t_1, \end{cases} \tag{7}$$

where $t_1 = (l/C_0)/(1 + (\alpha + 2u_0)/(8C_0))$. In (7), the profile distortion caused by elastoplastic deformation during propagation of the shock wave V_2 from the collision surface to the free surface of the impactor is ignored, but the phase shift due to nonlinearity is taken into account.

To describe the shock-wave evolution in the target, we find the solution V_1 of Eq. (1′) for the two-dimensional case ($\varepsilon_\Delta = 0$) subject to the initial condition

$$V_1 = \begin{cases} u_0/C_0 > 0, & -l \leqslant x_1 < 0, \\ 0, & 0 \leqslant x_1 \leqslant L, \end{cases} \qquad t = 0$$

and the boundary condition (7) with allowance for the fact that $V_2 = 0$ in the central cross sections of the target. The initial pressure was determined by solving the problem of the discontinuity decay at the contact boundary $x_1 = 0$. It was assumed that $\psi_1 = 2\tau/(3G)$, where $\tau = -(\sigma_1 - \sigma_2)/2$.

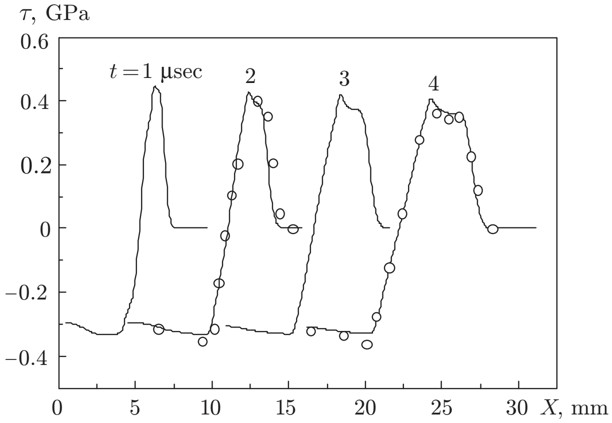
To describe the behavior of the medium, we employ the dislocation model. We use Eq. (6) in which ξ_1 is replaced by $\xi_1' = (x_1 - C_0 t)/l$ and determine the plastic strain rate by the Orowan relation [5]

$$\frac{\partial \varepsilon^{\rm pl}}{\partial \xi_1'} = bN_d v_d,$$

where b is the Burgers displacement. The specific dependences for the density of travelling dislocations N_d and their averaged velocities v_d were taken in accordance with the model proposed in [9]. The other parameters of the material were as follows: $\rho_0 = 2.787$ g/cm^3, bulk modulus $K = 764$ kbar, and Poisson's ratio was set equal to 0.33. The equation of state has the form $P = (K/n)[(\rho/\rho_0)^n - 1]$, where $n = 4.1$. Thus, the parameters α and $\nu' = 2\nu/(\alpha + 2)$ in Eq. (1′) take the values $\alpha = 3.1$ and $\nu' = 0.1$.

The problem posed was solved numerically using artificial viscosity for a collision velocity $u_0 = 1.2$ km/sec and an impactor thickness of 1.5 mm. The stress-wave profiles and the distribution of the maximum shear stresses τ in the target plate were obtained for various times after the impact. Figure 1 shows the curves $\tau(X)$, where $X = x_1|_{t=0}$. One can see that the results obtained agree with the numerical solution of the exact system of equations [9].

2.2. *Reflection of the Shock Pulse of Finite Duration from the Free Surface.* When the shock pulse arrives at the free surface, its reflection occurs, i.e., the incident pulse interacts with the reflected wave. Equations (1), (4), and (5) allow one to study the problem of shock-pulse reflection for small but finite strains.



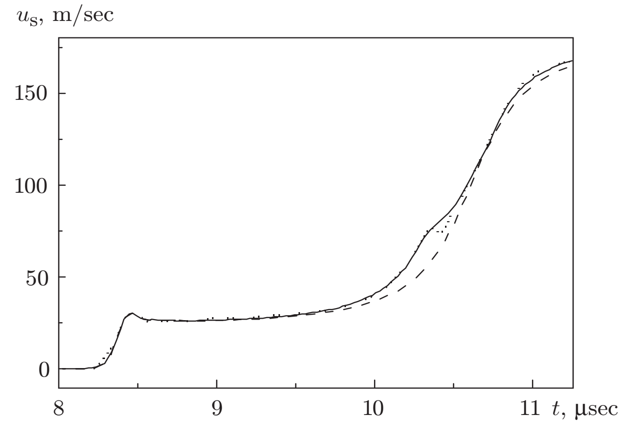


Fig. 1

Fig. 2

Fig. 1. Distribution of the shear stress τ at various times (the points refer to the numerical solution of [9]).

Fig. 2. Velocity of the free surface u_s versus time: the solid curve refers to the numerical solution that takes into account the interaction between the incident and reflected waves, the dashed curve refers to the numerical solution in which this interaction is ignored, and the dotted curve refers to the experimental data obtained by Taylor and Rice [10, p. 29].

We model the experimental data obtained by Taylor and Rice [10, p. 29] using the dislocation model [5]. The wave interaction was calculated using the one-dimensional variant of Eqs. (1) and equations for the phase functions (4) and (5). Figure 2 shows the velocity of the free surface u_s versus time for an armco-iron plate 5.08 cm thick, which was loaded by the normal impact with a velocity of 170 m/sec (the impactor was made of the same material as the target). One can see that the numerical solution that takes into account the interaction between the incident and reflected waves agrees well with the experimental data obtained by Taylor and Rice.

To measure the velocity of the specimen surface, we used a laser interferometer [10]. At the contact surface between the specimen and interferometer window, the intensity and sign of the reflected wave depend on the impedance ratio $\bar{æ} = \bar{\rho}\bar{C}/(\rho_0 C_0)$ ($\bar{\rho}$ and $\bar{C}$ are the density and speed of sound for the material of the window, respectively): $V_2 = -V_1(1 - \bar{æ})/(1 + \bar{æ})$. Windows of laser interferometers are usually made of quartz glass, sapphire, LiF, etc. If $\bar{æ} < 1$, the reflected wave is a rarefaction wave and its effect on the velocity of the contact surface of the specimen is similar to that in the case of the free surface $\bar{æ} = 0$ considered above. In this case, the amplitude of second step of the precursor depends on $\bar{æ}$. This conclusion is supported experimentally for the Ti–6Al–4V alloy [11], where a two-step precursor in the velocity profile of the specimen–LiF contact surface ($\bar{æ} < 1$) and a one-step precursor for the specimen–sapphire contact ($\bar{æ} > 1$) were determined.

3. Damage of a Plane Plate Produced by a Cylindrical Impactor. Using Eqs. (1), we solve numerically the model problem of the normal-impact damage of a plate of finite thickness, produced by a cylindrical impactor with a velocity of 185 m/sec. The phase shift caused by wave interaction was ignored. Equations (1) were solved by a numerical method similar to that used in [12].

The material of the impactor and target is aluminum with parameters $\rho_0 = 2.61$ g/cm^3, $C_0 = 5.3$ km/sec, and $C_{\rm long} = 6.4$ km/sec. As the constitutive equation, we use the equation that corresponds to an ideal-plastic body with the dynamic yield point $Y = 0.18$ GPa [13]. The impactor thickness is $l = 1.14$ mm and the target thickness is $L = 2.8l$. The radius of the impactor is $r_0 = 6l$. Hence, $\nu = 0.229$ and $\varepsilon_\Delta = 1/36$ in Eqs. (1). The characteristic time is $t_0 = l/C_0 = 0.215$ μsec. The initial and boundary conditions have the form

$$V_1(z = 0, \xi_1, r') = \varepsilon v(\xi_1) R(r'),$$

$$V_1\Big|_{\xi_1 \to -\infty} = \frac{\partial V_1}{\partial \xi_1}\Big|_{\xi_1 \to +\infty} = 0, \qquad V_1\Big|_{r' \to \infty} = \frac{\partial V_1}{\partial r'}\Big|_{r'=0} = 0.$$

Here

$$R(r') = \begin{cases} 1, & r' \leqslant 1, \\ 0, & r' > 1, \end{cases} \qquad v(\xi_1) = \begin{cases} 1, & \xi_1 \in [0, 2 - (\alpha + 2)u_0/(8C_0)], \\ 0, & \xi_1 \notin [0, 2 - (\alpha + 2)u_0/(8C_0)], \end{cases} \tag{8}$$

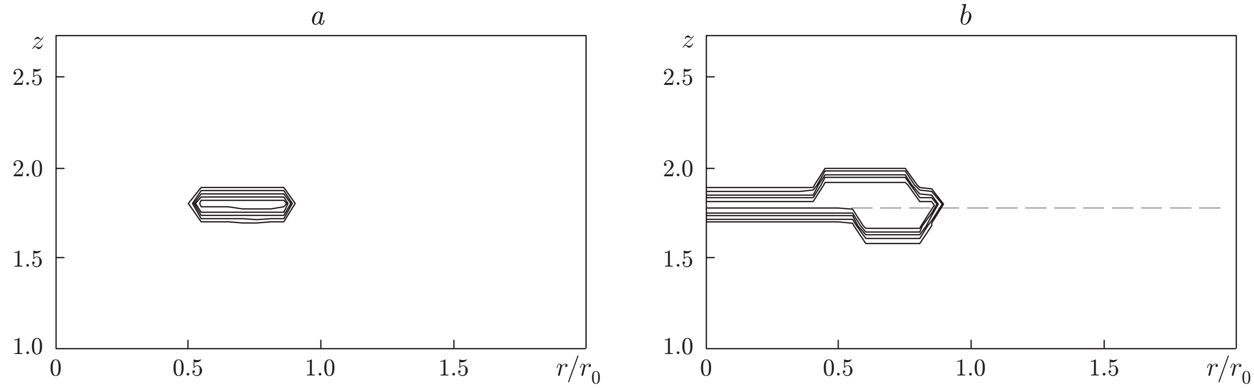


Fig. 3. Damage regions of the plate for $t = 3.92t_0$ (a) and $3.98t_0$ (b).

and the impact velocity is $u_0 = 185$ m/sec. The parameter $\varepsilon = 2P_m/K = 0.036$ (P_m is the stress amplitude). It follows from (8) that the collision of the plates occurs at the moment $\xi_1 = t = 0$. In (8), the effect of lateral unloading is ignored because the transverse dimensions of the impactor considerably exceed its thickness. At the free surface $z = L/l$, the boundary conditions $V_2 = -V_1$ and $\xi_2 = \xi_1$ are specified.

Equations governing the evolution of the material damage are taken the same as in [14]. The specific volume of microdefects ω is used as a damage measure. It is assumed that the material of the plate fails when the damage reaches the critical value. Figure 3 shows the calculation results for the two-dimensional damage of the plate (the impact is performed along the lower surface $0 \leqslant r/r_0 \leqslant 1$) for regions corresponding to $\omega \geqslant 0.01$. In Fig. 3b, the central region enclosed by the lines corresponds to the failed material. The calculations show that the minimum time it takes for the spall surface to form is $3.98t_0$. Failure begins under the impactor (Fig. 3a) and then propagates toward the center to form a disc-shaped crack (Fig. 3b). A similar result was obtained in [15, 16] by solving numerically the exact starting system of equations. In Fig. 3b, the dashed line shows the location of the spalling-fracture line predicted in [15] by solving a similar problem with close parameters of the material and impactor. One can see that the location of the spalling line agrees with our calculations.

In summary, the results obtained show that the model equations proposed describe qualitatively the stress-distribution evolution in the regions of elastic and plastic flow and can be used to solve one- and two-dimensional problems of pulsed deformation and fracture of elastoplastic media.

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